Simultaneous surds equation

1. Solve $\begin{cases} \sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}} = \frac{7}{\sqrt{xy}} \\ \sqrt[4]{x^3y} - \sqrt[4]{xy^3} = \sqrt{12} \end{cases}$.

$$\begin{cases} \sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}} = \frac{7}{\sqrt{xy}} \quad \ldots (1) \\ \sqrt[4]{x^3y} - \sqrt[4]{xy^3} = \sqrt{12} \ldots (2) \end{cases}$$

For $x, y > 0$, from (1), $x - y = 7 \ \ldots (3)$

From (2), $\sqrt[4]{xy}\left(\sqrt{x} - \sqrt{y}\right) = \sqrt{12}$

Squaring we get $\sqrt{\mathrm{xy}}\left(\sqrt{x} - \sqrt{y}\right)^2 = 12 \ \ldots (4)$

From (3), $\sqrt{x} - \sqrt{y} = \frac{7}{\sqrt{x}+\sqrt{y}} \ \ldots (5)$

(5) ↓ (4), $\sqrt{\mathrm{xy}}\left(\frac{7}{\sqrt{x}+\sqrt{y}}\right)^2 = 12$

$\Longrightarrow 49\sqrt{\mathrm{xy}} = 12\left(x + 2\sqrt{\mathrm{xy}} + y\right)$

$\Longrightarrow 12\mathrm{x} - 25\sqrt{\mathrm{xy}} + 12\mathrm{y} = 0$

$\Longrightarrow 12\left(\sqrt{\mathrm{x}}\right)^2 - 25\sqrt{\mathrm{xy}} + 12\left(\sqrt{\mathrm{y}}\right)^2 = 0$

$\Longrightarrow \left(4\sqrt{\mathrm{x}} - 3\sqrt{\mathrm{y}}\right)\left(3\sqrt{\mathrm{x}} - 4\sqrt{\mathrm{y}}\right) = 0$

$\Longrightarrow \sqrt{\frac{\mathrm{x}}{y}} = \frac{3}{4} \ or \ \frac{4}{3}$

$\Longrightarrow \frac{x}{y} = \frac{9}{16} \ or \ \frac{16}{9}$

$\Longrightarrow x = \frac{9}{16}y \ or \ x = \frac{16}{9}y \ \ldots (6)$

(6) ↓ (3), $\frac{9}{16}y - y = 7 \ or \ \frac{16}{9}y - y = 7$

$-\frac{7}{16}y = 7 \ or \ \frac{7}{9}y = 7$

$y = -16 \ or \ y = 9$

$(x, y) = (-9, -16) \ or \ (16,9)$.

Checking with (1) and (2), we have $(x, y) = (16,9)$.

Similarly, if $x, y < 0$, $(x, y) = (-16, -9)$.

Hence, $(x, y) = (-16, -9) \ or \ (16,9)$.

2. Solve $\begin{cases} \sqrt{7x+y}+\sqrt{x+y}=6 \\ \sqrt{x+y}-y+x=2 \end{cases}$.

$$\begin{cases} \sqrt{7x+y}+\sqrt{x+y}=6 \quad \ldots(1) \\ \sqrt{x+y}-y+x=2 \quad \ldots(2) \end{cases}$$

$(1)-(2)$, $\sqrt{7x+y}-y+x=x-4 \Longrightarrow \sqrt{7x+y}=4-y+x \quad \ldots(3)$

From (1),

$$\left(\sqrt{7x+y}+\sqrt{x+y}\right)\left(\sqrt{7x+y}-\sqrt{x+y}\right)=6\left(\sqrt{7x+y}-\sqrt{x+y}\right)$$
$$(7x+y)-(x+y)=6\left(\sqrt{7x+y}-\sqrt{x+y}\right)$$
$$6x=6\left(\sqrt{7x+y}-\sqrt{x+y}\right)$$
$$\sqrt{7x+y}-\sqrt{x+y}=x \quad \ldots(4)$$

$(4)+(2)$, $\sqrt{7x+y}-y+x=x+2$

$\Longrightarrow \sqrt{7x+y}=2+y \quad \ldots(5)$

$(3)=(5)$, $4-y+x=2+y$

$\Longrightarrow x=2y-2 \quad \ldots(6)$

$(6)\downarrow(5)$, $\sqrt{7(2y-2)+y}=2+y$

$\Longrightarrow \sqrt{15y-14}=2+y$

$\Longrightarrow 15y-14=4+4y+y^2$

$\therefore y^2-11y-18=0$

$\Longrightarrow (y-2)(y-9)=0$

$\Longrightarrow y=2,9$

When $y=9$, $x=16$ (rejected)

When $y=2$, $x=2$ (satisfies (1) and (2))

Solution : $(x,y)=(2,2)$.

Should also check : $(6)\downarrow(3)$. (Result is the same.)

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